Categorical Homotopy Type Theory

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UQÀM

MIT Topology Seminar, March 17, 2014

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- **Church**: *A formulation of the simple theory of types* (1940)
- **Lawvere**: *Equality in hyperdoctrines and comprehension schema as an adjoint functor* (1968)
- **Martin-Löf**: *Intuitionistic theory of types* (1971, 1975, 1984)
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- **Awodey, Warren**: *Homotopy theoretic models of identity types* (2006∼2007)
- **Voevodsky**: *Notes on type systems* (2006∼2009)

Suggested readings

Recent work in homotopy type theory
Slides of a talk by Steve Awodey at the AMS meeting January 2014

Notes on homotopy λ-calculus
Vladimir Voevodsky

Homotopy Type Theory
A book by the participants of the Univalent Foundation Program held at the IAS in 2012-13

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- equivalences $X \simeq Y$ by paths $X \rightsquigarrow Y$.

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Category theory as a bridge

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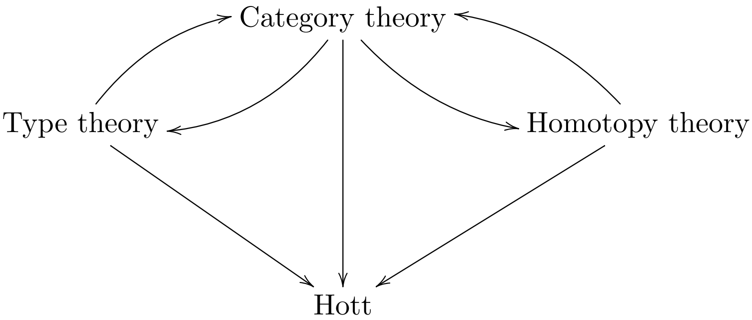
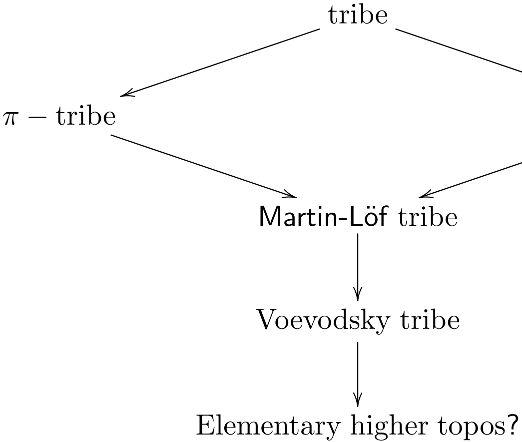


Overview of the talk



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$$\begin{array}{ccc} A \times_B X & \xrightarrow{\;p_2\;} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\;f\;} & B \end{array}$$

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The projection p_1 is called the **base change** of $p : X \to B$ *along* $f : A \to B$.

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A map in $\mathcal{F}$ is called a **fibration**.

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A tribe is a collection of families closed under certain operations.

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General contexts

Type declarations can be iterated:

$$A : Type$$

$$x : A \vdash B(x) : Type$$

$$x : A, y : B(x) \vdash C(x, y) : Type$$

$$x : A, y : B(x), z : C(x, y) \vdash E(x, y, z) : Type$$

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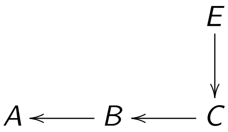

$\Gamma = (x : A, y : B(x), z : C(x, y))$ is a *general context*.

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Composition of maps is obtained by substituting:

$$\frac{x : A \vdash f(x) : B, \qquad y : B \vdash g(y) : C}{x : A \vdash g(f(x)) : C}$$

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Remark: The category of tribes is a 2-category if a 2-cell is a natural transformation.

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In type theory, it is expressed by the following *deduction rule*:

$$\frac{y : B \vdash E(y) : \mathit{Type}}{x : A \vdash E(f(x)) : \mathit{Type}.}$$

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In type theory, this is called a **context extension**.

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A term $t : \sum_{x:A} E(x)$ is a pair $t = (a, u)$, where $a : A$ and $u : E(a)$.

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(Gambino and Garner) The syntaxic category of type theory is a tribe, where a fibration is a map isomorphic to a display map

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Formally, we have

$$f_!(E)(y) = \sum_{f(x)=y} E(x).$$

for a term $y : B$.

Polynomial rings

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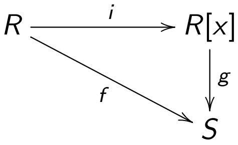

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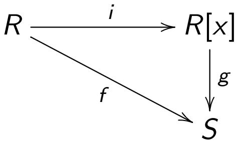

The element $x \in R[x]$ is **generic**.

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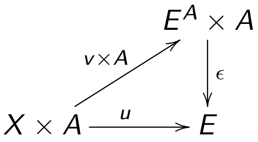

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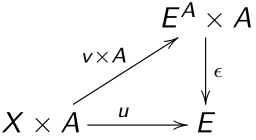

We write $v = \lambda^A(u)$.

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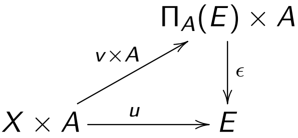

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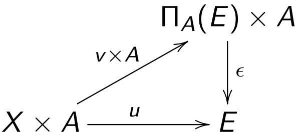

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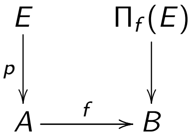
Products along a map

Let $f : A \to B$ be a quadrable map in a category $\mathcal{C}$.

The **product** $\Pi_f(E)$ of $E = (E, p) \in \mathcal{C}/A$ **along** $f : A \to B$ is the space of sections of the map $(E, fp) \to (A, f)$ in the category $\mathcal{C}/B$.



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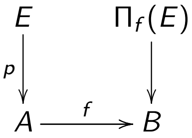

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We say that a tribe $\mathcal{C}$ is π-**closed**, and that it is a π-**tribe**, if every fibration $E \to A$ has a product along any fibration $f : A \to B$ and the structure map $\Pi_f(E) \to B$ is a fibration,

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If $\mathcal{C}$ is a π-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

Examples of π-tribes

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- A cartesian closed category, where a fibration is a projection;
- A locally cartesian category is a Π-tribe in which every map is a fibration;
- The category of small groupoids **Grpd**, where a fibration is an iso-fibration (Hofmann, Streicher);
- The category of Kan complexes **Kan**, where a fibrations is a Kan fibration (Voevodsky, Streicher);

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And a rule for the formation of λ-terms:

$$\frac{x : A \vdash t(x) : E(x)}{\vdash (\lambda x)t(x) : \prod_{x:A} E(x)}$$

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This means that every commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{b} & Y \end{array}$$

has a diagonal filler $d : B \to X$ ($du = a$ and $fd = b$).

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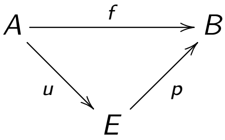

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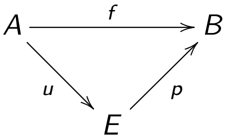

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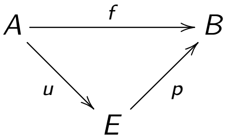

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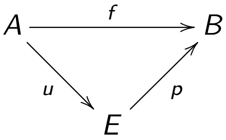

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If $\mathcal{C}$ is a h-tribe, then so is the tribe $\mathcal{C}(A)$ for every object $A \in \mathcal{C}$.

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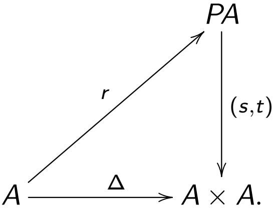
- The syntaxic category of Martin-Löf type theory, where a fibration is a map isomorphic to a display map (Gambino-Garner).

Path object

Path object

A **path object** for an object $A \in \mathcal{C}$ is a factorisation of the diagonal $\Delta : A \to A \times A$ as an anodyne map $r : A \to PA$ followed by a fibration $(s, t) : PA \to A \times A$,



Identity type

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There is a term

$$x : A \ \vdash r(x) : Id_A(x, x)$$

called the **reflexivity term**. It is a proof that $x = x$.

The J-rule

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There is an operation J which takes commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & E \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle p} \\ Id_A & = & Id_A \end{array}$$

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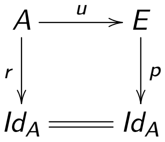
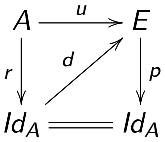


with p a fibration, to a diagonal filler $d = J(u, p)$



It shows that the reflexivity term $r : A \to Id_A$ is anodyne!

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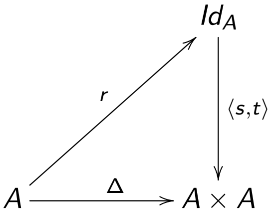

Mapping path space

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The **mapping path space** $P(f)$ of a map $f : A \to B$ is defined by the pullback square

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This gives a factorization $f = pu : A \to P(f) \to B$ with $u = \langle 1_A, rf \rangle$ an anodyne map and $p = tp_2$ a fibration.

The **homotopy fiber** of a map $f : A \to B$ at a point $y : B$ is the fiber of the fibration $p : P(f) \to B$ at the same point,

$$\mathrm{fib}_f(y) = \sum_{x:A} Id_A(f(x), y).$$

Homotopic maps

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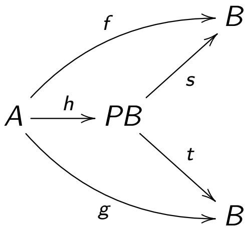

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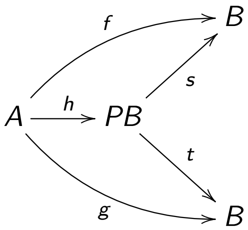

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An object X is **contractible** if the map $X \to \star$ is a homotopy equivalence.

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The inclusion $\mathcal{C}(A) \to \mathcal{C}/A$ induces an equivalence of categories:

$$Ho(\mathcal{C}(A)) = Ho(\mathcal{C}/A)$$

Homotopy pullback

A square

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

is a homotopy pullback if the canonical map $A \to B \times_D^h C$ is a homotopy equivalence, where $B \times_D^h C = (f \times g)^\star(PD)$

$$\begin{array}{ccc} B \times_D^h C & \longrightarrow & D \\ \downarrow & & \downarrow \\ B \times C & \xrightarrow{f \times g} & D \times D \end{array}$$

h-propositions

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A map $u : A \to B$ is *homotopy monic* if the square

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An object $A \in \mathcal{C}$ is a *h-proposition* if the map $A \to \star$ is homotopy monic.

An object A is a h-proposition if and only if the diagonal $A \to A \times A$ is a homotopy equivalence.

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Homotopy initial objects

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An object $\bot \in \mathcal{C}$ is **homotopy initial** if every fibration $p : E \to \bot$ has a section $\sigma : \bot \to E$,

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A homotopy initial object remains initial in the homotopy category $Ho(\mathcal{C})$.

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An object $A \sqcup B$ equipped with a pair of maps $i, j : A, B \to A \sqcup B$

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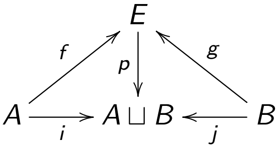

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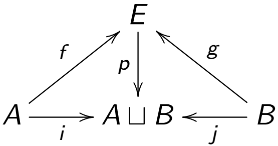

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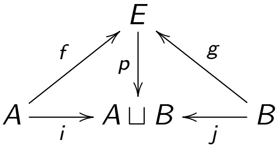

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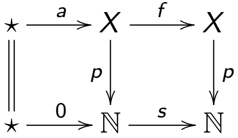

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A homotopy natural number object $(\mathbb{N}, s, 0)$ is not necessarily a natural number object in the homotopy category $Ho(\mathcal{C})$.

Martin-Löf tribe

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Definition

We say that a tribe $\mathcal{C} = (\mathcal{C}, \mathcal{F})$ is a **ML-tribe** if the following conditions are satisfied

- $\mathcal{C}$ is a Π-tribe and a *h*-tribe;

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If $\mathcal{C}$ is a ML-tribe, then so is the tribe $\mathcal{C}(A)$ for every $A \in \mathcal{C}$.

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It follows that there is a map

$$\prod_{x:A} Id_B(f(x), g(x)) \to Id_{[A,B]}(f, g)$$

defined for any pair of maps $f, g : A \to B$.

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Theorem

*(***Gambino-Garner***) The syntaxic category of Martin-Löf type theory, where a fibration is a map isomorphic to a display map.*

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A set S is *decidable* if the relations $x \in S$ and the equality relation $x = y$ for $x, y \in S$ can be decided recursively.

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Corollary

The syntaxic category of Martin-Löf type theory with homotopy natural numbers is decidable

Elementary topos

Let $\mathcal{E}$ be a category with finite limits

Recall that a monomorphism $t : 1 \to \Omega$ in $\mathcal{E}$ is said to be *universal* if for every monomorphism $S \to A$ there exists a unique map $f : A \to \Omega$, such that $f^{-1}(t) = S$,

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \downarrow & & \downarrow t \\ A & \xrightarrow{f} & \Omega \end{array}$$

The pair (Ω, t) is called a *sub-object classifier*.

Lawvere and Tierney:

Definition

An elementary topos is a locally cartesian category with a sub-object classifier Ω.

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A homotopical presheaf F is **representable** if the functor $F : Ho(\mathcal{C})^{op} \to \mathbf{Set}$ is representable.

$IsContr(X)$

If $E \in \mathcal{C}$, then the presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by putting

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Compare with

$$(\exists x \in E)\ (\forall y \in E)\ x = y$$

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If $f : X \to Y$ is an arbitrary map, then the presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ defined by putting

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where $\mathrm{fib}_f(y)$ is the homotopy fiber of f at $y : Y$.

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For every object $A \in \mathcal{C}$, there is a bijection between the maps

$$A \to Eq(X, Y)$$

in $Ho(\mathcal{C})$ and the isomorphism $X_A \simeq Y_A$ in $Ho(\mathcal{C}(A))$

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The identity $1_{E(x)}$ is represented by a term

$$x : A \vdash u(x) : Eq(E(x), E(x))$$

which defines the unit map $u : A \to Eq_A(E)$,

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$$
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PA & \xrightarrow{\simeq} & Eq_A(E) \\
 & {\scriptstyle \langle s,t \rangle} \searrow \quad \swarrow {\scriptstyle (s,t)} & \\
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\end{array}
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Uncompressible fibration

To *compress* a Kan fibration $p : X \to A$ is to find a homotopy pullback square

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Every Kan fibration $X \to A$ is the pullback of an uncompressible fibration $X' \to A'$ along a homotopy surjection $A \to A'$.

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Voevodsky's conjecture : The relations $\vdash t : A$ and $\vdash s = t : A$ are decidable in V-type theory.

What is an elementary higher topos?

Grothendieck topos	Elementary topos
Higher topos	EH-topos?

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Let $\mathcal{E}$ be a category with terminal object $\top$ and initial object $\bot$.

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A *generalized model structure* on $\mathcal{E}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of classes maps in $\mathcal{E}$ such that

- $\mathcal{W}$ satisfies 3-for-2;
- the pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems;
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- $\mathcal{W}$ satisfies 3-for-2;
- the pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems;
- the maps in $\mathcal{F}$ are quadrable and maps in $\mathcal{C}$ is co-quadrable;

A *generalized model category* is the category $\mathcal{E}$ equipped with a generalised model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

A map in $\mathcal{C}$ is called a *cofibration*, a map in $\mathcal{W}$ a *weak equivalence* and a map in $\mathcal{F}$ a *fibration*.

Generalised model categories

Let $\mathcal{E}$ be a category with terminal object $\top$ and initial object $\bot$.

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An object A is *cofibrant* if the map $\bot \to A$ is a cofibration, an object X is *fibrant* if the map $X \to \top$ is a fibration.

Remark : If every object of a generalized model category is cofibrant, then the class $\mathcal{F}$ determines the classes $\mathcal{C}$ and $\mathcal{W}$.

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A generalized model structure is **smooth** if it is excellent, π-closed and every object is cofibrant.

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Critic 3: We may need to suppose that every fibration factors as a homotopy surjection followed by a fibrant monomorphism.

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THANK YOU!