Transformation of Sensitivities

Christopher L. P. Wolfe*

July 10, 2020

1 Introduction

We often want adjoint sensitivities with respect to variables that are combinations of variables for which we already have sensitivities. For example, we commonly have the sensitivities of a cost function, $\mathcal{J}$, to wind stress,

$$\frac{\partial \mathcal{J}}{\partial \tau^x} \quad \text{and} \quad \frac{\partial \mathcal{J}}{\partial \tau^y}, \tag{1}$$

but want the sensitivity to wind stress curl,

$$\frac{\partial \mathcal{J}}{\partial \varpi}, \tag{2}$$

where $\varpi = \operatorname{curl} \boldsymbol{\tau}$. Simply taking the curl of the sensitivities to wind stress doesn't produce a result with the right units: if the units of $\mathcal{J}$ are j, then the sensitivity to wind stress, $\partial_\tau \mathcal{J}$, has units of j N^{-1} m^2. The units of the sensitivity to wind stress curl, $\partial_\varpi \mathcal{J}$ are j N^{-1} m^3, but the units of

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{J}}{\partial \tau^y} - \frac{\partial}{\partial y}\frac{\partial \mathcal{J}}{\partial \tau^x} \tag{3}$$

are j N^{-1} m and so are off by a factor of the area.

The proper way to do the transformation is to use the chain rule. This is straightforward in 1D, but not so much in 2D. The main problem in 2D is that the problem seems to require the specification of boundary conditions, but it's not clear what those boundary conditions should be. (Boundary conditions are required in 1D, but it's more obvious what they should be.)

Here we start with the 1D warm up and then proceed to the 2D case.

2 1D warm up

2.1 Discrete version

Suppose we have the sensitivities of a cost function with respect to wind stress, τ_i, on an $N + 1$ point grid; that is, we have

$$\frac{\partial \mathcal{J}}{\partial \tau_i}. \tag{4}$$

We want the sensitivity to the derivative of the wind stress τ'_j, where the prime indicates differentiation. Using the chain rule, we have

$$\frac{\partial \mathcal{J}}{\partial \tau'_j} = \sum_{i=0}^{N} \frac{\partial \tau_i}{\partial \tau'_j}\frac{\partial \mathcal{J}}{\partial \tau_i}. \tag{5}$$

*christopher.wolfe@stonybrook.edu

We need the τ in terms of the τ' to evaluate the elements of the transformation matrix.

On a C-grid,

$$\tau'_j = \frac{\tau_j - \tau_{j-1}}{\Delta x}, \tag{6}$$

from which it follows that

$$\tau_i = \tau_0 + \Delta x \sum_{k=0}^{i-1} \tau'_k, \tag{7}$$

so

$$\frac{\partial \tau_i}{\partial \tau'_j} = \Delta x \sum_{k=0}^{i-1} \delta_{jk} = \begin{cases} \Delta x & j < i, \\ 0 & j \geq i. \end{cases} \tag{8}$$

The transformed sensitivity is therefore

$$\frac{\partial \mathcal{J}}{\partial \tau'_j} = \Delta x \sum_{i=j+1}^{N} \frac{\partial \mathcal{J}}{\partial \tau_i} \tag{9}$$

2.2 Continuum version

We can do this in the continuous limit as well. By the chain rule,

$$\frac{\partial \mathcal{J}}{\partial \tau'(x')} = \int_0^L \frac{\delta \tau(x)}{\delta \tau'(x')} \frac{\partial \mathcal{J}}{\partial \tau(x)} \, \mathrm{d}x, \tag{10}$$

where

$$\frac{\delta \tau(x)}{\delta \tau'(x')} \tag{11}$$

is the functional derivative of $\tau(x)$ with respect to $\tau'(x')$. Note that in 1D, the function derivative brings in an additional unit of inverse length which is canceled by the integration.

We can integrate to get the stress in terms of its derivative:

$$\tau(x) = \tau_0 + \int_0^x \tau'(x'') \, \mathrm{d}x''. \tag{12}$$

The functional derivative of τ wrt τ' is

$$\frac{\delta \tau(x)}{\delta \tau'(x')} = \int_0^x \frac{\delta \tau'(x'')}{\delta \tau'(x')} \, \mathrm{d}x'' = \int_0^L \delta(x'' - x') \, \mathrm{d}x'' = \mathcal{H}(x - x'), \tag{13}$$

where $\mathcal{H}$ is the Heaviside step function and we have used the fact that

$$\frac{\delta f(x)}{\delta f(x')} = \delta(x - x') \tag{14}$$

for any function f. The transformed sensitivity is therefore

$$\frac{\partial \mathcal{J}}{\partial \tau'(x')} = \int_{x'}^L \frac{\partial \mathcal{J}}{\partial \tau(x)} \, \mathrm{d}x, \tag{15}$$

and so is related to the original sensitivity by integration.

2.3 2D problem

Now suppose we have the sensitivities of a cost function with respect to wind stress as a function of 2D location $\boldsymbol{x}$; that is, we have

$$\frac{\partial \mathcal{J}}{\partial \tau^x(\boldsymbol{x})} \quad \text{and} \quad \frac{\partial \mathcal{J}}{\partial \tau^y(\boldsymbol{x})}, \tag{16}$$

and we want the sensitivity to the wind stress curl ϖ. Again, by the chain rule,

$$\frac{\partial \mathcal{J}}{\partial \varpi(\boldsymbol{x})} = \int_{\mathcal{D}} \left[\frac{\delta \tau^x(\boldsymbol{x}')}{\delta \varpi(\boldsymbol{x})} \frac{\partial \mathcal{J}}{\partial \tau^x(\boldsymbol{x}')} + \frac{\delta \tau^y(\boldsymbol{x}')}{\delta \varpi(\boldsymbol{x})} \frac{\partial \mathcal{J}}{\partial \tau^y(\boldsymbol{x}')} \right] \mathrm{d}A', \tag{17}$$

where the integration is over the 2D area $\mathcal{D}$ occupied by the ocean. We need to write the wind field in terms of its curl to actually evaluate this. If we have both the curl and the divergence, Δ, of the wind field, we can solve the problems

$$\nabla^2 \phi = -\Delta, \tag{18}$$

$$\nabla^2 \psi = \varpi, \tag{19}$$

and recover the wind field from

$$\boldsymbol{\tau} = -\nabla \phi + \hat{\boldsymbol{k}} \times \nabla \psi. \tag{20}$$

The fields ϕ and ψ must satisfy the consistency conditions

$$\oint_{\partial \mathcal{D}} \nabla \phi \cdot \hat{\boldsymbol{n}} \, \mathrm{d}s = -\oint_{\partial \mathcal{D}} \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}s, \tag{21}$$

$$\oint_{\partial \mathcal{D}} \nabla \psi \cdot \hat{\boldsymbol{n}} \, \mathrm{d}s = -\oint_{\partial \mathcal{D}} \left(\hat{\boldsymbol{k}} \times \boldsymbol{\tau} \right) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}s. \tag{22}$$

This is all very straightforward in a periodic domain, since we don't need to worry about boundary conditions in that case. If the domain is not periodic, we need to pick boundary conditions, and it's not clear what those should be.

Formal solutions are available in terms of Green's functions

$$\phi = -\int_{\mathcal{D}} G_\phi(\boldsymbol{x} - \boldsymbol{x}') \Delta(\boldsymbol{x}') \, \mathrm{d}A' - \oint_{\partial \mathcal{D}} \left[G_\phi(\boldsymbol{x} - \boldsymbol{x}') \nabla' \phi(\boldsymbol{x}') \cdot \hat{\boldsymbol{n}} - \phi(\boldsymbol{x}') \nabla' G_\phi(\boldsymbol{x} - \boldsymbol{x}') \cdot \hat{\boldsymbol{n}} \right] \mathrm{d}s', \tag{23}$$

$$\psi = \int_{\mathcal{D}} G_\psi(\boldsymbol{x} - \boldsymbol{x}') \varpi(\boldsymbol{x}') \, A' - \oint_{\partial \mathcal{D}} \left[G_\psi(\boldsymbol{x} - \boldsymbol{x}') \nabla' \psi(\boldsymbol{x}') \cdot \hat{\boldsymbol{n}} - \psi(\boldsymbol{x}') \nabla' G_\psi(\boldsymbol{x} - \boldsymbol{x}') \cdot \hat{\boldsymbol{n}} \right] \mathrm{d}s', \tag{24}$$

where G_ϕ and G_ψ are the Green's functions for ϕ and ψ, respectively, and ∇' indicates the derivative with respect to the primed variables. The Green's functions satisfy same differential equation,

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{25}$$

but may have different boundary conditions, depending on the boundary conditions on ϕ and ψ.

2.4 Periodic domains

In a periodic domain, no boundary conditions are needed. We need a slightly modified formulation of the Green's function which can be derived by multiplying the Poisson equation by G and integrating over the domain:

$$\int_{\mathscr{D}} G\nabla^2\psi \,\mathrm{d}A = \int_{\mathscr{D}} \varpi G \,\mathrm{d}A. \tag{26}$$

Making use of Green's theorem gives

$$\int_{\mathscr{D}} \psi\nabla^2 G \,\mathrm{d}A = \int_{\mathscr{D}} \varpi G \,\mathrm{d}A. \tag{27}$$

Let the Green's function satisfy

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}') - \frac{1}{\mathscr{A}}, \tag{28}$$

where $\mathscr{A}$ is the area of the domain $\mathscr{D}$. Removing the inverse area is necessary to satisfy the condition

$$\int \nabla^2 G \,\mathrm{d}A = 0. \tag{29}$$

The formal solution becomes

$$\psi(\boldsymbol{x}) = \int_{\mathscr{D}} G(\boldsymbol{x} - \boldsymbol{x}')\varpi(\boldsymbol{x}') \,\mathrm{d}A' + \frac{1}{\mathscr{A}} \int_{\mathscr{D}} \psi \,\mathrm{d}A. \tag{30}$$

We can eliminate this last term by demanding that ψ have zero integral. The formal solutions for the two potentials is therefore

$$\phi = -\int_{\mathscr{D}} G(\boldsymbol{x} - \boldsymbol{x}')\Delta(\boldsymbol{x}') \,\mathrm{d}A', \tag{31}$$

$$\psi = \int_{\mathscr{D}} G(\boldsymbol{x} - \boldsymbol{x}')\varpi(\boldsymbol{x}') \,\mathrm{d}A', \tag{32}$$

and the wind stress is

$$\boldsymbol{\tau}(\boldsymbol{x}) = \int_{\mathscr{D}} \left[\nabla G(\boldsymbol{x} - \boldsymbol{x}'')\Delta(\boldsymbol{x}'') + \hat{\boldsymbol{k}} \times \nabla G(\boldsymbol{x} - \boldsymbol{x}'')\varpi(\boldsymbol{x}'')\right] \mathrm{d}A''. \tag{33}$$

The functional derivative of $\boldsymbol{\tau}$ with respect to ϖ is therefore

$$\begin{aligned} \frac{\delta\boldsymbol{\tau}(\boldsymbol{x})}{\delta\varpi(\boldsymbol{x}')} &= \int_{\mathscr{D}} \hat{\boldsymbol{k}} \times \nabla G(\boldsymbol{x} - \boldsymbol{x}'')\frac{\delta\varpi(\boldsymbol{x}'')}{\delta\varpi(\boldsymbol{x}')} \,\mathrm{d}A'' \\ &= \int_{\mathscr{D}} \hat{\boldsymbol{k}} \times \nabla G(\boldsymbol{x} - \boldsymbol{x}'')\delta(\boldsymbol{x}'' - \boldsymbol{x}') \,\mathrm{d}A'' \\ &= \hat{\boldsymbol{k}} \times \nabla G(\boldsymbol{x} - \boldsymbol{x}') \\ &= -\hat{\boldsymbol{k}} \times \nabla' G(\boldsymbol{x} - \boldsymbol{x}'). \end{aligned} \tag{34}$$

In Cartesian coordinates, the components of the transformation matrix are

$$\frac{\delta\tau^x(\boldsymbol{x})}{\delta\varpi(\boldsymbol{x}')} = \frac{\partial}{\partial y'}G(\boldsymbol{x} - \boldsymbol{x}'), \tag{35}$$

$$\frac{\delta\tau^y(\boldsymbol{x})}{\delta\varpi(\boldsymbol{x}')} = -\frac{\partial}{\partial x'}G(\boldsymbol{x} - \boldsymbol{x}'), \tag{36}$$

and the sensitivity to the wind stress curl is therefore

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial \varpi(\boldsymbol{x})} &= \int_{\mathcal{D}} \left[\frac{\partial}{\partial y'} G(\boldsymbol{x}-\boldsymbol{x}') \frac{\partial \mathcal{J}}{\partial \tau^x(\boldsymbol{x}')} - \frac{\partial}{\partial x'} G(\boldsymbol{x}-\boldsymbol{x}') \frac{\partial \mathcal{J}}{\partial \tau^y(\boldsymbol{x}')} \right] \mathrm{d}x' \, \mathrm{d}y' \\ &= \int_{\mathcal{D}} G(\boldsymbol{x}-\boldsymbol{x}') \left[\frac{\partial}{\partial x} \frac{\partial \mathcal{J}}{\partial \tau^y(\boldsymbol{x}')} - \frac{\partial}{\partial y'} \frac{\partial \mathcal{J}}{\partial \tau^x(\boldsymbol{x}')} \right] \mathrm{d}x' \, \mathrm{d}y', \end{aligned} \tag{37}$$

where the second equality follows from integration-by-parts.

In the end, you do the curl of the sensitivities, but then convolve the result with the Green's function for the Poisson equation.