Transformation of Sensitivities

Christopher L. P. Wolfe*

July 7, 2020

Suppose we have the sensitivities of a cost function with respect to wind stress, τ_i; that is, we have

$$\frac{\partial \mathcal{J}}{\partial \tau_i}, \tag{1}$$

and we want the sensitivity to the derivative of the wind stress τ'_j. Using the chain rule, we have

$$\frac{\partial \mathcal{J}}{\partial \tau'_j} = \sum_{i=0}^{N} \frac{\partial \tau_i}{\partial \tau'_j} \frac{\partial \mathcal{J}}{\partial \tau_i}. \tag{2}$$

On a C-grid,

$$\tau'_j = \frac{\tau_j - \tau_{j-1}}{\Delta x}, \tag{3}$$

from which it follows that

$$\tau_i = \tau_0 + \Delta x \sum_{k=0}^{i-1} \tau'_k, \tag{4}$$

so

$$\frac{\partial \tau_i}{\partial \tau'_j} = \Delta x \sum_{k=0}^{i-1} \delta_{jk} = \begin{cases} \Delta x & j < i, \\ 0 & j \geq i. \end{cases} \tag{5}$$

The transformed sensitivity is therefore

$$\frac{\partial \mathcal{J}}{\partial \tau'_j} = \Delta x \sum_{i=j+1}^{N} \frac{\partial \mathcal{J}}{\partial \tau_i} \tag{6}$$

We can do this in the continuous limit as well. By the chain rule,

$$\frac{\delta \mathcal{J}}{\delta \tau'(x')} = \int_0^L \frac{\delta \tau(x)}{\delta \tau'(x')} \frac{\delta \mathcal{J}}{\delta \tau(x)} \, \mathrm{d}x. \tag{7}$$

We integrate to get the stress in terms of its derivative:

$$\tau(x) = \tau_0 + \int_0^x \tau'(x'') \, \mathrm{d}x''. \tag{8}$$

The functional derivative of τ wrt τ' is

$$\frac{\delta \tau(x)}{\delta \tau'(x')} = \int_0^x \frac{\delta \tau'(x'')}{\delta \tau'(x')} \, \mathrm{d}x'' = \int_0^L \delta(x'' - x') \, \mathrm{d}x'' = \mathcal{H}(x - x'), \tag{9}$$

where $\mathcal{H}$ is the Heaviside step function. The transformed sensitivity is therefore

$$\frac{\delta \mathcal{J}}{\delta \tau'(x')} = \int_{x'}^L \frac{\delta \mathcal{J}}{\delta \tau(x)} \, \mathrm{d}x. \tag{10}$$

*christopher.wolfe@stonybrook.edu